APPENDIX

The new feature encountered in evaluating the asymptotic behavior of (17) is due to the fact that putting β_1 equal to zero in the factor Δ_1 in the denominator gives a δ_1 integration which diverges at $\delta_1=0$. In order to evaluate correctly the asymptotic form of (17), it is necessary therefore to integrate in the neighborhood of $\beta_1=\cdots=\beta_n=\delta_2=0$ *and* $\delta_1=0$. In order to evaluate the leading asymptotic behavior it is only necessary to consider the linear terms in Δ_1. The structure of Δ_1 is such that these terms only involve δ_1 and β_1. Thus, the leading behavior can be obtained by evaluating

$$\int_0^\epsilon \frac{d\beta d\delta_1 d\delta_2}{[c_1\delta_1+B\beta_1][c_2\beta_1\cdots\beta_n\delta_2 S+d]^r}. \tag{A1}$$

The δ_1 integration is performed first to give

$$-\frac{1}{c_1}\int_0^\epsilon \frac{d\beta d\delta_2[\ln B\beta_1-\ln(c_1\epsilon+B\beta_1)]}{[c_2\beta_1\cdots\beta_n\delta_2 S+d]^r}. \tag{A2}$$

The second term in the numerator of (A2) is bounded when $\beta_1=0$ so does not contribute to the leading asymptotic behavior. It will, therefore, be omitted. The β_1 integration is now performed and yields

$$\frac{1}{(r-1)c_1c_2d^{r-1}}\int_0^\epsilon \frac{d\beta_2\cdots d\beta_n d\delta_2}{\beta_2\cdots\beta_n\delta_2 S} \times[\ln(\epsilon\beta_2\cdots\beta_n\delta_2 S+1)+O(1)] \tag{A3}$$

$$\sim\frac{1}{(r-1)d^{r-1}}\cdot\frac{1}{c_1c_2}\cdot\frac{(\ln S)^{n+1}}{S\Gamma(n+2)},\quad S\to\infty. \tag{A4}$$

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Low-Energy $\bar{K}$–d Scattering*

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The magnitude of recoil and binding effects in the multiple-scattering corrections to the impulse approximation in low-energy $\bar{K}$-d scattering is examined by the introduction of a model which makes tractable the numerical evaluation of the double-scattering terms. The finite mass of the nucleons and the n-p interaction in continuum states are both taken into account. It is concluded that estimates of multiple-scattering corrections which ignore these effects are not reliable. The model is used to compute the sum of the cross sections for $K^-+d\to K^-+d$, $K^-+d\to K^-+p+n$. Comparison with the rather limited data available in the region 100 to 200 MeV/c favors the so-called solution II found by Humphrey and Ross in their analysis of $\bar{K}$-p data based on the Dalitz scattering lengths. A pseudopotential or optical-model-like approach to meson-deuteron scattering, which may be useful in other problems, is also described.

1. INTRODUCTION

SOME years ago, Dalitz[1] introduced two complex scattering lengths, which seem adequate for the phenomenological description of low-energy $\bar{K},N$ scattering and absorption processes. Considerable ambiguity in the values of A_0 and A_1, the $I=0$ and $I=1$ scattering lengths, respectively, was allowed by the data, and a number of attempts were made to reduce the ambiguity by a comparison of the rather limited data on K^--d reactions with theoretical predictions.[2] The present work was begun in an attempt to estimate the validity of previous calculations and to improve them, if possible.

More recently, the work of Ross and Humphrey[3] narrowed the ambiguity to a choice of two solutions, so-called solutions I and II, corresponding, respectively, to

I: $A_0=-0.22+2.74i$ F, $A_1=0.02+0.38i$ F,

II: $A_0=-0.59+0.96i$ F, $A_1=1.2+0.56i$ F.

Akiba and Capps[4] then showed that only solution II is consistent with the data of Tripp *et al.*[5] obtained in the reaction $K^-+p\to\Sigma+\pi$ at 400 MeV/c.

We may, thus, turn the problem around and ask to what extent an analysis of K^--d scattering processes supports this choice, or better, to what extent one may correctly predict K^--d scattering and reaction cross sections, using this choice of the phenomenological scattering lengths.

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[1] For a review, see R. H. Dalitz, *Strange Particles and Strong Interactions* (Oxford University Press, London, 1962).
[2] T. B. Day, G. A. Snow, and J. Sucher, Nuovo Cimento **14**, 637 (1959); Phys. Rev. **119**, 1110 (1960).
[3] W. R. Humphrey and R. R. Ross, University of California Radiation Laboratory Reports UCRL-9749 and UCRL-9752 (unpublished).
[4] T. Akiba and R. H. Capps, Phys. Rev. Letters **8**, 457 (1962).
[5] R. Tripp, M. Watson, and M. Ferro-Luzzi, Phys. Rev. Letters **8**, 175 (1962); **9**, 28 (1962).

In the following section, a model for the scattering of a particle from a two-particle bound state is introduced which permits numerical evaluation of double-scattering effects with inclusion of binding interactions in intermediate states, as shown in Sec. 3. In Sec. 4, the relation of this model to the Brueckner model[6] is studied and a modified form of this latter model is thereby suggested. Numerical results for both models are compared and it is shown that reliable estimates of double-scattering corrections are not likely to be obtainable in any model which neglects recoil and binding in intermediate states, such as the Brueckner model. The limited comparison possible with the present model and the available data is made in Sec. 5, and a concluding discussion is given in the final section.

Although we do not go beyond double-scattering terms in the body of this paper, it is possible to include those multiple-scattering effects, of arbitrarily high order, in which the deuteron is always in its ground state while the meson is scattering from one nucleon to the other. The summation of the corresponding infinite series is achieved by the introduction of relatively simple pseudopotentials, analogous to the optical potentials used to describe scattering from a nucleus consisting of A nucleons. However, a "$1/A$" approximation is *not* made, nor are the nucleons assumed to be identical. The resulting formula is a generalization of one due to Francis and Watson, and the methods employed may be useful in other contexts. These matters are described in the Appendix.

2. FORMALISM AND MODEL

The transition operator for scattering of a particle "0" (meson) by a composite particle (deuteron) consisting of particles "1" and "2" (nucleons) is given, in the meson-deuteron c.m. system, by

$$t=V+V(E-K-V-V_{12}+i\epsilon)^{-1}V\,, \tag{2.1}$$

where V_{ij} denotes the (effective) interaction between particles i and j, $V=V_{01}+V_{02}$ is the meson-deuteron interaction, and K and E are the kinetic-energy operator and energy, respectively, in the c.m. system. The matrix element of interest is then

$$M=M(\mathbf{k}',\mathbf{k})=\langle\mathbf{k}',\varphi|t|\mathbf{k},\varphi\rangle\,, \tag{2.2}$$

where

$$|\mathbf{k},\varphi\rangle=e^{i\mathbf{k}\cdot\mathbf{r}}\varphi(\boldsymbol{\rho})\,, \tag{2.3}$$

with $\mathbf{k}$ and $\mathbf{k}'$ the initial and final momenta of the meson in the c.m. system, $\mathbf{r}$ the meson coordinate with respect to the deuteron center of mass, $\boldsymbol{\rho}$ the relative proton-neutron coordinate, and $\varphi(\boldsymbol{\rho})$ the wave function of the deuteron ground state. Thus, with $\mathbf{r}_i$ the (laboratory) coordinate and m_i the mass of the particle "i", we have

$$\mathbf{r}=\mathbf{r}_0-(\alpha_1\mathbf{r}_1+\alpha_2\mathbf{r}_2),\quad \boldsymbol{\rho}=\mathbf{r}_1-\mathbf{r}_2\,,$$

where

$$\alpha_i=m_i/(m_1+m_2)\approx\tfrac{1}{2}\,.$$

The multiple-scattering expansion of t is given by[7]

$$t=\sum_{i=1}^{2}t_i+\sum_{i\neq j}^{2}t_iGt_j+\sum_{i\neq j,\ j\neq k}^{2}t_iGt_jGt_k+\cdots\,, \tag{2.4}$$

where t_i is defined, implicitly, by

$$t_i=V_{0i}+V_{0i}Gt_i\quad(i=1,2) \tag{2.5}$$

and

$$G=(E-K-V_{12}+i\epsilon)^{-1}. \tag{2.6}$$

Correspondingly,

$$M=M_1+M_2+M_{12}+M_{21}+\cdots\,, \tag{2.7}$$

where

$$M_i=\langle\mathbf{k}',\varphi|t_i|\mathbf{k},\varphi\rangle \tag{2.8}$$

and

$$M_{ij}=\langle\mathbf{k}',\varphi|t_iGt_j|\mathbf{k},\varphi\rangle\quad(i\neq j)\,. \tag{2.9}$$

In Eq. (2.7) only the terms that will be considered explicitly in this paper are exhibited.

The model in question is defined in the first instance by the replacement

$$\text{(i)}\quad t_1\to t_1^0\delta(\mathbf{r}_{01}),\quad t_2\to t_2^0\delta(\mathbf{r}_{02}),\quad(\mathbf{r}_{0i}=\mathbf{r}_0-\mathbf{r}_i)\,, \tag{2.10}$$

in Eq. (2.7), where the t_i^0 are constants, which will, however, in any application, be allowed to vary with the energy of the incident meson. This replacement corresponds to treating the nucleons as heavy, noninteracting scatterers, while the meson is scattering repeatedly from *only one* of the nucleons, via a very short-range interaction. By taking the t_i^0 to be proportional to the free meson-nucleon scattering amplitudes, the first two terms of Eq. (2.7) reduce to the usual "simple impulse approximation."[8]

The effect of recoil and binding corrections to the multiple-scattering terms will be studied by assuming a form for the p-n interaction which will make the numerical evaluation of the double-scattering terms tractable, without any further assumptions or approximations. This is achieved by choosing a V_{12} which permits the determination of the continuum p-n states in closed form and yields the Hulthén wave function for the single bound state. Thus, we take for V_{12} the nonlocal but separable potential[9]

$$\text{(ii)}\quad V_{12}(\boldsymbol{\rho},\boldsymbol{\rho}')=-\beta(\alpha+\beta)^2(4\pi\mu_{12}\rho\rho')^{-1}e^{-\beta\rho}e^{-\beta\rho'}. \tag{2.11}$$

The continuum wave functions corresponding to asymptotic relative p-n momentum $\mathbf{q}$ are then easily found:

$$\psi_{\mathbf{q}}^{(+)}(\boldsymbol{\rho})=\exp(i\mathbf{q}\cdot\boldsymbol{\rho})+f(q)F_q(\boldsymbol{\rho})\,,$$

[6] K. A. Brueckner, Phys. Rev. **89**, 834 (1953).

[7] K. M. Watson, Phys. Rev. **89**, 575 (1953).

[8] The nucleon spin is being ignored and only an S-wave meson-nucleon interaction is allowed for, since we have in mind applications to $\bar{K}$ meson-nucleon interactions at low energies, and not π mesons, where even at the energies considered here, the P-wave interaction is important.

[9] Y. Yamaguchi, Phys. Rev. **95**, 1628 (1954).

where

$$f(q)=\left[\frac{(q^2+\beta^2)^2}{2\beta(\alpha+\beta)^2}+\frac{q^2-\beta^2}{2\beta}-iq\right]^{-1}$$

and

$$F_q(\varrho)=(e^{iq\rho}-e^{-\beta\rho})/\rho.$$

The bound-state wave function is

$$\varphi(\varrho)=N(e^{-\alpha\rho}-e^{-\beta\rho})/\rho,\quad [N^2=\alpha\beta(\alpha+\beta)/2\pi(\alpha+\beta)^2]$$

with Fourier transform

$$\chi(\mathbf{p})=(2\pi)^{-3}\int\exp(-i\mathbf{p}\cdot\varrho)\varphi(\varrho)d\varrho$$
$$=(N/2\pi^2)[(\mathbf{p}^2+\alpha^2)^{-1}-(\mathbf{p}^2+\beta^2)^{-1}].$$

To fit the deuteron binding energy and the triplet p-n scattering length, we must take

$$\alpha=45.6\text{ MeV}/c\quad\text{and}\quad\beta=6.2\alpha.$$

We also note that since V_{12} is Hermitian, we have the completeness relation

$$\int\frac{d\mathbf{l}}{(2\pi)^3}\left[\int\frac{d\mathbf{q}}{(2\pi)^3}|\mathbf{l},\psi_{\mathbf{q}}^{(+)}\rangle\langle\mathbf{l},\psi_{\mathbf{q}}^{(+)}|+|\mathbf{l},\varphi\rangle\langle\mathbf{l},\varphi|\right]=1. \tag{2.12}$$

Equations (2.4), (2.6), (2.10), and (2.11) define our model completely. The relation of this model to the Brueckner model[6] will be discussed in Sec. 4.

We may relate t_1^0 and t_2^0 to the S-wave meson-nucleon scattering amplitudes f_1 and f_2, by considering the scattering of the meson from a nucleon. The scattering amplitude is given by

$$f_j=-(\mu_{0j}/2\pi)\langle\boldsymbol{\kappa}'|t_j|\boldsymbol{\kappa}\rangle,\quad (j=1,2), \tag{2.13}$$

where $\boldsymbol{\kappa}$ and $\boldsymbol{\kappa}'$ are the initial and final meson momenta in the meson-nucleon c.m. system and $\mu_{0j}=(m_0m_j)/(m_0+m_j)$ is the reduced mass.

From Eq. (2.11) we then get

$$f_j=-(\mu_{0j}/2\pi)t_j^0. \tag{2.14}$$

3. COMPUTATION

With assumptions (i) and (ii) [Eqs. (2.10) and (2.11)], the matrix element M reduces to

$$M=M_{\text{imp}}+M_d+\cdots, \tag{3.1}$$

where

$$M_{\text{imp}}=(t_1^0+t_2^0)S((\mathbf{k}'-\mathbf{k})/2), \tag{3.2}$$

and

$$S(\mathbf{k})=\int\exp(-i\mathbf{k}\cdot\varrho)\varphi^2(\varrho)d\varrho \tag{3.3}$$

is the deuteron form factor. M_d denotes the contribution of the double-scattering terms and may be written as

$$M_d=M_d^B+M_d^C, \tag{3.4}$$

where M_d^B and M_d^C denote, respectively, the contribution of the deuteron ground state and deuteron continuum states to the double-scattering process. Thus, using Eqs. (2.6), (2.9), and (2.12), we get

$$M_d^B=2t_1^0t_2^0\int\frac{d\mathbf{l}}{(2\pi)^3}D_E^{-1}(\mathbf{l})\langle\mathbf{k}',\varphi|\delta(\mathbf{r}-\varrho/2)|\mathbf{l},\varphi\rangle\times\langle\mathbf{l},\varphi|\delta(\mathbf{r}+\varrho/2)|\mathbf{k},\varphi\rangle \tag{3.5}$$

and

$$M_d^C=2t_1^0t_2^0\int\frac{d\mathbf{l}d\mathbf{q}}{(2\pi)^6}D_E^{-1}(\mathbf{l},\mathbf{q})\times\langle\mathbf{k}',\varphi|\delta(\mathbf{r}-\varrho/2)|\mathbf{l},\psi_{\mathbf{q}}^{(+)}\rangle\times\langle\mathbf{l},\psi_{\mathbf{q}}^{(+)}|\delta(\mathbf{r}+\varrho/2)|\mathbf{k},\varphi\rangle, \tag{3.6}$$

where

$$D_E(\mathbf{l})=E-\mathbf{l}^2/2\mu-\epsilon_B+i\epsilon,$$
$$D_E(\mathbf{l},\mathbf{q})=E-\mathbf{l}^2/2\mu-\mathbf{q}^2/2\mu_{12}+i\epsilon,$$

with μ and μ_{12} the reduced meson-deuteron and nucleon-nucleon masses, respectively, $E=(\mathbf{k}^2/2\mu)+\epsilon_B$, and ϵ_B the deuteron binding energy.

The factor of 2 arises from the equal contribution of the third and fourth terms in Eq. (2.7).

The factors in the numerator in Eq. (3.5) reduce to

$$\langle\mathbf{k}',\varphi|\delta(\mathbf{r}-\varrho/2)|\mathbf{l},\varphi\rangle=S((\mathbf{k}'-\mathbf{l})/2),$$

and, similarly,

$$\langle\mathbf{l},\varphi|\delta(\mathbf{r}+\varrho/2)|\mathbf{k},\varphi\rangle=S((\mathbf{l}-\mathbf{k})/2).$$

For Eq. (3.6), we need

$$\langle\mathbf{k}',\varphi|\delta(\mathbf{r}-\varrho/2)|\mathbf{l},\psi_{\mathbf{q}}^{(+)}\rangle=\int\exp[-i(\mathbf{k}'-\mathbf{l})\cdot\varrho/2]\varphi(\varrho)\psi_{\mathbf{q}}^{(+)}(\varrho)d\varrho$$
$$=(2\pi)^3\chi(\tfrac{1}{2}(\mathbf{k}'-\mathbf{l})-\mathbf{q})+f(q)\mathfrak{F}(\mathbf{k}',\mathbf{l};q),$$

where

$$\mathfrak{F}(\mathbf{k}',\mathbf{l};q)=\langle\mathbf{k}',\varphi|\delta(\mathbf{r}-\varrho/2)|\mathbf{l},F_q(\varrho)\rangle,$$

and, similarly,

$$\langle\mathbf{l},\psi_{\mathbf{q}}^{(+)}|\delta(\mathbf{r}+\varrho/2)|\mathbf{k},\varphi\rangle=(2\pi)^3\chi(\tfrac{1}{2}(\mathbf{l}-\mathbf{k})+\mathbf{q})+f^*(q)\mathfrak{F}^*(\mathbf{k},\mathbf{l};q).$$

In general, the resulting six-dimensional integrals may be reduced to triple integrals which could be evaluated by the use of a high-speed computing machine. For forward scattering, all integrals may be reduced to double integrals, amenable to desk computation. We confine ourselves to this case here. Thus, we put $\mathbf{k}'=\mathbf{k}$ and obtain

$$M_d^B(k)\equiv M_d^B(\mathbf{k},\mathbf{k})=4\mu t_1^0t_2^0 \times\int\frac{d\mathbf{l}}{(2\pi)^3}(\mathbf{k}^2-\mathbf{l}^2+i\epsilon)^{-1}|S((\mathbf{k}-\mathbf{l})/2)|^2 \quad (3.7)$$

and

$$M_d^C(k)\equiv M_d^C(\mathbf{k},\mathbf{k})=\sum_{j=1}^{4} M^{(j)}(k), \quad (3.8)$$

with

$$M^{(1)}(k)=2t_1^0t_2^0\int\int d\mathbf{l}d\mathbf{q}\; D_E^{-1}(\mathbf{l},\mathbf{q}) \times|\chi(\tfrac{1}{2}(\mathbf{k}-\mathbf{l})-\mathbf{q})|^2,$$

$$M^{(2)}(k)=2t_1^0t_2^0\int\int \frac{d\mathbf{l}d\mathbf{q}}{(2\pi)^3}D_E^{-1}(\mathbf{l},\mathbf{q}) \times f(q)\chi(\tfrac{1}{2}(\mathbf{k}-\mathbf{l})-\mathbf{q})\mathfrak{F}(\mathbf{k},\mathbf{l};q), \quad (3.9)$$

$$M^{(3)}(k)=[M^{(2)}(k)]^*,$$

$$M^{(4)}(k)=2t_1^0t_2^0\int\int \frac{d\mathbf{l}d\mathbf{q}}{(2\pi)^6}{}_BD_E^{-1}(\mathbf{l},\mathbf{q}) \times|f(q)\mathfrak{F}(\mathbf{k},\mathbf{l};q)|^2.$$

$M^{(1)}(k)$ corresponds to the plane-wave approximation for the continuum states and the other terms represent the effect of binding interactions in these states. With the use of Feynman's method for combining denominators, and some other standard tricks, the integrations over $\mathbf{l}$ and $\mathbf{q}$ may be carried out. The integration over the auxiliary variables was then done numerically.[10]

4. CORRESPONDENCE WITH THE BRUECKNER MODEL

A. Relation Between the Models

We may relate the Brueckner model[6] to our model as follows. The $K^- - d$ scattering amplitude f is related to the transition amplitude M by

$$f=f(\mathbf{k}',\mathbf{k})=-(\mu/2\pi)M, \quad (4.1)$$

where μ is the meson-deuteron reduced mass and M is given by Eq. (2.2). Thus,

$$f=-(\mu/2\pi)\langle\mathbf{k}',\varphi|t_1+t_2+t_1Gt_2 +t_2Gt_1+\cdots|\mathbf{k},\varphi\rangle. \quad (4.2)$$

Using Eq. (2.10), we get on integrating now first over $\mathbf{r}$, rather than $\boldsymbol{\varrho}$,

$$\langle\mathbf{k}',\varphi|t_1|\mathbf{k},\varphi\rangle=\langle\varphi|t_1^0\exp[i(\mathbf{k}-\mathbf{k}')\cdot\boldsymbol{\varrho}/2]|\varphi\rangle, \quad (4.3\text{a})$$

and

$$\langle\mathbf{k}',\varphi|t_2|\mathbf{k},\varphi\rangle=\langle\varphi|t_2^0\exp[-i(\mathbf{k}-\mathbf{k}')\cdot\boldsymbol{\varrho}/2]|\varphi\rangle. \quad (4.3\text{b})$$

[10] See A. K. Bhatia, Ph.D. thesis, University of Maryland, 1962, for further details. (Available as Technical Report No. 265.)

The double-scattering contributions may be written in the form

$$\langle\mathbf{k}^1,\varphi|t_1G_1t_2|\mathbf{k},p\rangle =\int\frac{d\mathbf{l}}{(2\pi)^3}\sum_n\frac{\langle\mathbf{k}',\varphi|t_1|\mathbf{l},n\rangle\langle\mathbf{l},n|t_2|\mathbf{k},\varphi\rangle}{(k^2/2\mu)+\epsilon_B-(\mathbf{l}^2/2\mu)-E_n+i\epsilon}, \quad (4.4)$$

where $\sum_n$ denotes a sum over a complete set of eigenstates, $\{|n\rangle\}$, of h_d, the deuteron Hamiltonian, with E_n the eigenvalues.

Let us suppose that E_n is replaced by some average value $\bar{E}$ and define $\bar{k}$ by

$$\bar{k}^2/2\mu=k^2/2\mu+\epsilon_B-\bar{E}. \quad (4.5)$$

Using Eq. (4.5) in Eq. (4.4), and the completeness relation

$$\sum_n|n\rangle\langle n|=\delta(\boldsymbol{\varrho}-\boldsymbol{\varrho}'),$$

as well as the relation,

$$\int\frac{d\mathbf{l}}{(2\pi)^3}\frac{|\mathbf{l}\rangle\langle\mathbf{l}|}{(\bar{k}^2/2\mu)-(\mathbf{l}^2/2\mu)+i\epsilon}=\frac{-\mu}{2\pi}\frac{\exp(i\bar{k}|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|},$$

we find, using Eq. (2.10) for t_1 and t_2, that

$$\langle\mathbf{k}',\varphi|t_1Gt_2|\mathbf{k},\varphi\rangle\to-(\mu/2\pi)t_1^0t_2^0 \times\langle\varphi|\exp[i(\mathbf{k}+\mathbf{k}')\cdot\boldsymbol{\varrho}/2]\exp(i\bar{k}\rho)\rho^{-1}|\varphi\rangle. \quad (4.6\text{a})$$

Similarly,

$$\langle\mathbf{k}',\varphi|t_2Gt_1|\mathbf{k},\varphi\rangle\to-(\mu/2\pi)t_1^0t_2^0 \times\langle\varphi|\exp[-i(\mathbf{k}+\mathbf{k}')\cdot\boldsymbol{\varrho}/2]\exp(i\bar{k}\rho)\rho^{-1}|\varphi\rangle. \quad (4.6\text{b})$$

Substituting Eqs. (4.3a), (4.3b), (4.4a) and (4.4b) in Eq. (4.2), we get

$$f\to\langle\varphi|\eta_1\exp[-i(\mathbf{k}'-\mathbf{k})\cdot\boldsymbol{\varrho}/2]+\eta_2 \times\exp[+i(\mathbf{k}'-\mathbf{k})\cdot\boldsymbol{\varrho}/2]+\eta_1\eta_2\exp(i\bar{k}\rho)\rho^{-1} \times\{\exp[i(\mathbf{k}'+\mathbf{k})\cdot\boldsymbol{\varrho}/2]+\exp[-i(\mathbf{k}'+\mathbf{k})\cdot\boldsymbol{\varrho}/2]\} +\cdots|\varphi\rangle, \quad (4.7)$$

where we define η_j by

$$\eta_j=(\mu/\mu_{0j})f_j,$$

and use has been made of Eq. (2.14).

The Brueckner model for the problem at hand takes the form

$$f^{\mathrm{Br}}=\langle\varphi|F(\mathbf{k}',\mathbf{k};\boldsymbol{\varrho})|\varphi\rangle,$$

where $F(\mathbf{k}',\mathbf{k};\boldsymbol{\varrho})$ is the amplitude for the meson to scatter from two infinitely heavy point nucleons with separation $\boldsymbol{\varrho}$:

$$F(\mathbf{k}',\mathbf{k};\boldsymbol{\varrho})=\{\zeta_1\exp[-i(\mathbf{k}'-\mathbf{k})\cdot\boldsymbol{\varrho}/2] +\zeta_2\exp[+i(\mathbf{k}'-\mathbf{k})\cdot\boldsymbol{\varrho}/2]+\zeta_1\zeta_2e^{ik\rho}\rho^{-1} \times(\exp[i(\mathbf{k}'+\mathbf{k})\cdot\boldsymbol{\varrho}/2] +\exp[-i(\mathbf{k}'+\mathbf{k})\cdot\boldsymbol{\varrho}/2])\} \times(1-\zeta_1\zeta_2e^{2ik\rho}\rho^{-2})^{-1}. \quad (4.8)$$

Here, ζ_j represents a meson-nucleon S-wave scattering amplitude for an infinitely heavy nucleon. For nucleons with finite mass some choice of ζ_j must be made to relate them to the measurable scattering amplitudes f_j.

On expanding the denominator in Eq. (4.8) in powers of $\zeta_1\zeta_2$ we get

$$f^{\mathrm{Br}}=\langle\varphi|\zeta_1\exp[-i(\mathbf{k}'-\mathbf{k})\cdot\boldsymbol{\rho}/2]+\zeta_2\exp[i(\mathbf{k}'-\mathbf{k})\cdot\boldsymbol{\rho}/2]+\zeta_1\zeta_2 e^{ik\rho}\rho^{-1}\times(\exp[i(\mathbf{k}+\mathbf{k}')\cdot\boldsymbol{\rho}/2]+\exp[-i(\mathbf{k}'+\mathbf{k})\cdot\boldsymbol{\rho}/2])+\cdots|\varphi\rangle. \quad (4.9)$$

We see that Eq. (4.7) and Eq. (4.9) are similar in form and the first two terms on the right-hand side of Eq. (4.7) and Eq. (4.8) (impulse approximation) will be the same provided we make the identification

$$\zeta_j=\eta_j.$$

The double-scattering terms will then also be the same provided we choose $\bar{k}=k$. Moreover, it is easy to verify that, if in higher order terms of Eq. (4.2) we make the corresponding replacements of energy denominators by average values and always choose $\bar{k}=k$, the resulting series will be identical to the series whose leading terms are shown in Eq. (4.9), after the replacement $\zeta_j\rightarrow\eta_j$.

We thus have made contact with the Brueckner model and at the same time motivated the choice of $\zeta_j=\eta_j$ used in a previous calculation[2] of K^--d scattering based on the Brueckner model. (One is also led to this choice by a consideration of the limit of weak meson-nucleon interaction and comparison with the impulse approximation.) Our derivation suggests that the Brueckner model approximates the effects of the continuum states very badly, since $\bar{k}=k$ implies the choice $\bar{E}=+\epsilon_B<0$! This statement is supported by the numerical results reported below. It seems likely that some choice of $\bar{k}$ such that $\bar{k}\neq k$ might give better agreement with the results of our model calculation. This would be interesting to pursue in a further investigation.

B. Comparison of Double-Scattering Corrections

The double-scattering effects in our model and in the Brueckner model may be compared independently of the values of η_1 and η_2. We denote the contributions to f and f^{Br} by f_d and f_d^{Br}, respectively. Then

$$f_d=-(\mu/2\pi)M_d=-(\mu/2\pi)(M_d^B+M_d^C)=f_d^B+f_d^C, \quad (4.10)$$

and

$$f_d^{\mathrm{Br}}=2\eta_1\eta_2\langle\varphi|e^{ik\rho}\rho^{-1}\sin[(\mathbf{k}+\mathbf{k}')\cdot\boldsymbol{\rho}/2]|\varphi\rangle. \quad (4.11)$$

f_d^{Br} may be obtained in closed form for the Hulthén wave function. The integrals defining M_d^B, the bound-state contribution [Eq. (3.7)] and M_d^C, the continuum-state contribution [Eq. (3.9)] have been evaluated numerically for forward scattering ($\mathbf{k}'=\mathbf{k}$). The results are given in Table I for the lab momentum $k_L=0$, 105, and 194 MeV/c.

TABLE I. Comparison of the double-scattering contributions to the forward scattering amplitude for $K^-+d\rightarrow K^-+d$. The second and third columns represent the bound state and bound-state plus continuum-state contributions, respectively, in the present model [Eq. (4.10)], while the last column is the contribution obtained from the Brueckner model [Eq. (4.11)].

k_L (MeV/c)	$f_d^B/2\eta_1\eta_2$	$f_d/2\eta_1\eta_2$	$f_d^{\mathrm{Br}}/2\eta_1\eta_2$
0	$0.93+0.00i$	$0.45+0.00i$	$0.57+0.00i$
105	$0.56+0.44i$	$-0.07+0.55i$	$0.38+0.24i$
194	$0.36+0.47i$	$-0.15+0.79i$	$0.21+0.27i$

We conclude, on comparing the third and fourth columns of Table I, that the Brueckner model is quite unreliable for $k_L\lesssim 200$ MeV/c, especially for what would be the real part of f_d if η_1 and η_2 were real. Comparison with the second column shows that in some sense the Brueckner model approximates the effects of the continuum states very badly, as discussed in the previous section, agreeing much better with just the bound-state contribution. This result seems reasonable from a physical point of view.

5. K^- MESON-DEUTERON FORWARD SCATTERING AMPLITUDE

In this section we shall give numerical results for the forward elastic K^-d scattering amplitude $f(0)$ based on our model, including the effects of charge exchange as well as recoil and binding effects in double scattering, but neglecting triple or higher order multiple scattering. As mentioned before, triple integrations are required for finding $f(\theta)$ for $\theta\neq 0$, even in our highly simplified model. These would require machine calculation and have not been carried out. In order to make some contact with experiment, we use the optical theorem to compute the total cross section σ_T, via

$$\sigma_T=4\pi k^{-1}\,\mathrm{Im}f(0).$$

Unfortunately, σ_T is not known, experimentally, at the low energies in question. However, there are data available on the total *nonabsorptive* cross section, $\sigma_{\mathrm{n.a.}}$, the sum of elastic and "breakup" cross sections:

$$\sigma_{\mathrm{n.a.}}=\sigma(K^-+d\rightarrow K^-+d)+\sigma(K^-+d\rightarrow K^-+p+n).$$

We now make a crude estimate for σ_a, the total absorptive cross section (hyperon or $\bar{K}^0$ production processes) via

$$\sigma_a\sim\sigma_a^p+\sigma_a^n, \quad (5.1)$$

where σ_a^p and σ_a^n denote the absorption cross section for K^- mesons incident on free protons and neutrons, respectively. The Dalitz scattering lengths of course have been determined so as to reproduce, among other quantities, the experimental values of σ_a^p, and σ_a^n may be computed from the Dalitz lengths and the assumed charge independence of the $(\bar{K},n)$ interaction. Since

$$\sigma_{\mathrm{n.a.}}=\sigma_T-\sigma_a,$$

TABLE II. Comparison of experimental and theoretical values for σ', the sum of the cross sections for $K^- + d \rightarrow K^- + d$ and $K^- + d \rightarrow K^- + n + p$, for the choices I and II [Eqs. (1.1) and (1.2)] of the $\bar{K}-N$ scattering lengths.

k_L (MeV/c)	$\sigma_{\text{exp.}}'$(mb)[a]	$\sigma_{\text{th.}}'$(mb) (Sol. I)	$\sigma_{\text{th.}}'$(mb) (Sol. II)
105		55	148
125	145±35		
175	55±15		
194		23	44

[a] See L. Alvarez, in *Proceedings of the Ninth International Annual Conference on High Energy Physics* (Academy of Sciences, U.S.S.R., 1960), p. 471.

we may compare experimental values of $\sigma_{\text{n.a.}}$ with the quantity σ':

$$\sigma' = 4\pi k^{-1} \operatorname{Im} f(0) - \sigma_a{}^p - \sigma_a{}^n.$$

We now also include in the computation of $f(0)$ the effects of virtual charge exchange,

$$K^- + d \rightarrow \bar{K}^0 + n + n \rightarrow K^- + d.$$

Neglecting the $n-n$ interaction, we find[10]

$$M_{12}{}^{\text{ex}} \approx (t_1{}^{0(\text{ex})} t_2{}^{0(\text{ex})} / 2 t_2{}^0 t_2{}^0) M^{(1)}(k),$$

where $M^{(1)}(k)$ [Eq. (3.9)] denotes the contribution of the continuum (p,n) states to M_d, in the plane-wave approximation, and $t_1{}^{0(\text{ex})}$, $t_2{}^{0(\text{ex})}$ are the (equal) transition amplitudes for $K^- + p \rightarrow \bar{K}^0 + n$ and $\bar{K}^0 + n \rightarrow K^- + p$.

Using Eq. (2.14) and the relations

$$f_p = \tfrac{1}{2}[A_1/(1-ikA_1) - A_0/(1-ikA_0)],$$
$$f_n = A_1/(1-ikA_1),$$
$$f_{\text{ex}} = \tfrac{1}{2}[A_1/(1-ikA_1) - A_0/(1-ikA_0)],$$

we find the results shown in Table II.[11]

6. CONCLUDING DISCUSSION

While a study of Table II shows that solution II is favored by the data, it should be emphasized that the estimate of Eq. (5.1) made to get this comparison is rather crude. A more serious test and application of the model in question would require the computations for other angles than 0° of the elastic amplitude, including a more careful estimate of the virtual charge exchange—and this could certainly be done by machine calculation—as well as a direct computation of the breakup cross section $\sigma(K^- + d \rightarrow K^- + p + n)$, within the framework of this model. Again, this seems feasible and the results obtained here would seem to justify further work along this line. Our conclusion that solution II is favored by the deuterium data is the same as that reached recently by Chand and Dalitz,[12] who have studied the same problem by an entirely different method which is, however, closely related to the Brueckner model, binding and recoil corrections in intermediate states being neglected. Our results (Table I) for the double-scattering part of the amplitude in a model in which such corrections have been included do not support the conjecture of Chand[13] that corrections of this kind are unlikely to be important.

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APPENDIX: PSEUDOPOTENTIAL METHODS IN MESON-DEUTERON SCATTERING

We consider here the summation of an infinite series which occurs in a study of the elastic scattering of an incident particle ("meson") by a bound system ("nucleus") consisting of A particles, ("nucleons") not necessarily identical.[14] The series is obtained by replacing the Green's function G, which occurs in the multiple-scattering expansion of the transition operator t, by GP where $P = |\varphi\rangle\langle\varphi|$ is the projection operator onto the ground state of the nucleus, with wave function φ. This corresponds to the so-called "coherent" contribution. If only these terms are included, and if a "$1/A$" approximation is made, the series is then effectively summed by introducing the operator $\sum_{i=1}^{A} \langle\varphi|t_i|\varphi\rangle$, the first approximation to an optical potential. We now show that the series may be summed without making such an approximation, especially when $A=2$.

The scattering amplitude f is given by

$$f = -(\mu/2\pi)\langle \mathbf{k}', \varphi | t | \mathbf{k}, \varphi \rangle,$$

where

$$t = \sum t_i + \sum_{i \neq j} t_i G t_j + \sum_{i \neq j,\, j \neq k} t_i G t_j G t_k + \cdots, \tag{1}$$

all considered in the over-all c.m. system, with μ the reduced meson-nucleus mass. If we denote by a bar the reduction of a many-body operator to an operator which acts only on the relative coordinate $\mathbf{r}$ of the meson with respect to the center of mass of the nucleus by taking

[11] For the purposes of this relatively crude comparison with the data, it was not thought worthwhile to include mass difference and Coulomb effects.

[12] R. Chand and R. H. Dalitz, Ann. Phys. (to be published).

[13] R. Chand, Ph.D. thesis, University of Chicago, 1962 (unpublished).

[14] This Appendix is based on unpublished work by one of the authors (J.S.).

the expectation value with respect to φ, we may write

$$f=-(\mu/2\pi)\langle\mathbf{k}'|\bar{t}|\mathbf{k}\rangle,$$

with

$$\bar{t}=\langle\varphi|t|\varphi\rangle.$$

On writing

$$G=GP+G(1-P)$$
$$=G_0P+G',$$

where

$$P=|\varphi\rangle\langle\varphi|,\quad G_0=(k^2/2\mu-\mathbf{p}^2/2\mu+i\epsilon)^{-1},$$

with $\mathbf{p}=-i\partial/\partial\mathbf{r}$, we get

$$\bar{t}=\bar{t}^B+\bar{t}',$$

where

$$\bar{t}^B=\sum\bar{t}_i+\sum_{i\neq j}\bar{t}_iG_0\bar{t}_j+\sum_{i\neq j,\ j\neq k}\bar{t}_iG_0\bar{t}_jG_0\bar{t}_k+\cdots,\quad(2)$$

and

$$\bar{t}'=\sum_{i\neq j}\langle\varphi|t_iG't_j|\varphi\rangle+\sum_{i\neq j,\ j\neq k}\langle\varphi|t_iG't_jG_0Pt_k+\cdots|\varphi\rangle+\cdots.\quad(3)$$

Here

$$\bar{t}_i=\langle\varphi|t_i|\varphi\rangle.\quad(4)$$

Correspondingly,

$$f=f^B+f',\quad(5)$$

with

$$f^B=-(\mu/2\pi)\langle\mathbf{k}'|\bar{t}^B|\mathbf{k}\rangle\quad(6)$$

and

$$f'=-(\mu/2\pi)\langle\mathbf{k}'|\bar{t}'|\mathbf{k}\rangle.\quad(7)$$

Now let $T=T[U]$ denote the transition operator for a meson moving in an arbitrary "external" potential U (a linear operator on functions of $\mathbf{r}$). Thus,

$$T=U+UG_0T=U(1-G_0U)^{-1}$$
$$=U+UG_0U+UG_0UG_0U+\cdots,\quad(8)$$

and the corresponding scattering amplitude is given, on the one hand, by

$$f[U]=-(\mu/2\pi)\langle\mathbf{k}'|T[U]|\mathbf{k}\rangle,$$

but, on the other, by solution of

$$(\mathbf{p}^2/2\mu+U)\chi(\mathbf{r})=(k^2/2\mu)\chi(\mathbf{r}),\quad(9a)$$

subject to the boundary condition

$$\chi(\mathbf{r})\sim\exp[i\mathbf{k}\cdot\mathbf{r}]+fe^{ikr}/r.\quad(9b)$$

The quantity of interest is f^B, defined by Eq. (6). If in Eq. (2), we include the terms with $i=j$, etc., we get

$$\bar{t}^B=v+vG_0v+vG_0vG_0v+\cdots+O(1/A),$$

where $v=\sum\bar{t}_i$, so that

$$\bar{t}^B\approx T[v]$$

and solution of Eqs. (9a, b), with $U=v$, will yield f^B correct to terms of order $1/A$. For $A=2$, this is not useful.

We now show that f^B may be evaluated without making a $1/A$ approximation for the case $A=2$, of interest for deuteron scattering. We also consider, with $A=2$, first a special case, where the $\bar{t}_i$ differ only in strength. In the case of $K^- -d$ scattering, this is the case, for example, if the δ-function model, Eq. (2.10), is used for the t_i. For then

$$\bar{t}_1\rightarrow t_1^0|\varphi(\mathbf{r}/2)|^2,\quad\bar{t}_2\rightarrow t_2^0|\varphi(\mathbf{r}/2)|^2.$$

A. *A = 2; special case.*

If the $\bar{t}_i$ have the same "shape" we may put

$$\bar{t}_i=\lambda_i\tau.\quad(i=1,2).\quad(10)$$

If we define λ as the geometric mean of λ_1 and λ_2, and $\bar{\lambda}$ as the arithmetic mean,

$$\lambda=(\lambda_1\lambda_2)^{1/2},\quad\bar{\lambda}=\tfrac{1}{2}(\lambda_1+\lambda_2),$$

then the series, Eq. (2), for $\bar{t}^B$ may be rewritten as follows:

$$\bar{t}^B=(2\bar{\lambda}/\lambda)(\lambda\tau+\lambda^3\tau G_0\tau G_0\tau+\cdots)+2(\lambda^2\tau G_0\tau+\lambda^4\tau G_0\tau G_0\tau G_0\tau+\cdots),$$

or, using Eq. (8),

$$\bar{t}^B=(\bar{\lambda}/\lambda)(T[\lambda\tau]-T[-\lambda\tau])+(T[\lambda\tau]+T[-\lambda\tau]).\quad(11)$$

Hence, if $f_\pm$ denotes the result of solving Eq. (9a, b), with the pseudopotential $U=\pm\lambda\tau$, i.e.,

$$f_\pm=-(\mu/2\pi)\langle\mathbf{k}'|T[\pm\lambda\tau]|\mathbf{k}\rangle,$$

we may write f^B as a linear combination of f_+ and f_-:

$$f^B=(\bar{\lambda}/\lambda)(f_+-f_-)+(f_++f_-),$$

or

$$f^B=(1+\bar{\lambda}/\lambda)f_++(1-\bar{\lambda}/\lambda)f_-.\quad(12)$$

Thus, for example, with the model introduced in Sec. 2, the higher order effects under study here may be included by taking $\lambda_i=t_i^0$, solving the Schrödinger equation with the local potentials $\pm(t_1^0t_2^0)^{1/2}|\varphi(\mathbf{r}/2)|^2$ to find $f_\pm$, and using Eq. (12) to find f^B.

We note that if $\bar{t}_1=\bar{t}_2$, then with $\lambda_1=\lambda_2=1$, $f^B\rightarrow 2f_+$. From this point of view, Eq. (12) constitutes a generalization of a formula given by Francis and Watson.[15]

B. *A = 2; general case.*

We define a set of "odd" and "even" pseudopotentials, V_i^o and V_i^e by

$$V_1^o=\bar{t}_1,\quad V_2^o=\bar{t}_2$$
$$V_1^e=\bar{t}_2G_0\bar{t}_1,\quad V_2^e=\bar{t}_1G_0\bar{t}_2.\quad(13)$$

V_i^o corresponds to single scattering by particle "i", and V_i^e corresponds to double scattering, starting on "i". The words "single" and "double" refer here to the multiple-scattering expansion. Using Eqs. (2), (8), and

[15] N. C. Francis and K. M. Watson, Phys. Rev. **92**, 291 (1953).

(13), we find that

$$t^B = V_1^o(1+G_0T[V_1^e]) + V_2^o(1+G_0T[V_2^e]) + T[V_1^e] + T[V_2^e]. \quad (14)$$

We now define scattering wave functions $\chi_i^e(\mathbf{r})$ and amplitudes f_i^e by

$$(\mathbf{p}^2/2\mu + V_i^e - \mathbf{k}^2/2\mu)\chi_i^e(\mathbf{r}) = 0, \quad (15a)$$

with

$$\chi_i^e(\mathbf{r}) \sim \exp(i\mathbf{k}\cdot\mathbf{r}) + f_i^e e^{ikr}/r. \quad (15b)$$

It follows that

$$f^B = -(\mu/2\pi)\langle\mathbf{k}'|t^B|\mathbf{k}\rangle = \sum_{i=1}^{2}(f_i^o + f_i^e), \quad (16)$$

where

$$f_i^o = -(\mu/2\pi)\langle\mathbf{k}'|V_i^o|\chi_i^e\rangle, \quad (17)$$

and the relations

$$\chi_i^e = (1+G_0T[V_i^e])|\mathbf{k}\rangle$$

and

$$f_i^e = -(\mu/2\pi)\langle\mathbf{k}'|T[V_i^e]/\mathbf{k}\rangle$$

have been used.

The decomposition of f^B described by Eq. (16) has a simple physical interpretation. f_i^e and f_i^o correspond, respectively, to the sum of all odd and the sum of all even number of scatterings, starting on particle "i" f_i^e may be computed by solving Eq. (15a, b) with the pseudopotential V_i^e. The wave function χ_i^e, obtained in the course of this computation, may then be used to compute f_i^o, via Eq. (17).

The problem of the evaluation of f^B has thus again, as in A above, been reduced to the solution of two single-particle Schrödinger equations. This time, however, the potentials appearing in these equations are nonlocal, even if the t_i are local. Nevertheless, for low-incident energies, where a partial-wave expansion of χ_i^e converges rapidly, the resulting integrodifferential equations for the radial wave function $u_l(r)$ has a relatively simple form and may be solved numerically.

The discussion of the special case in A shows that if Eq. (10) holds, this nonlocality may be avoided altogether. The point is that then f^B, the sum of the terms in Eq. (16), is more easily computed than the individual terms, and the rearrangement of the series for t^B indicated in Eq. (14) is not profitable; the appropriate rearrangement is then given by Eq. (11).

Finally, we remark that these methods and results may be generalized to $A>2$, at least if Eq. 10 holds. This is shown in C below.

C. $A>2$; special case:

If we define partial transition operators implicitly by

$$\tau_i = t_i + \sum_{j\neq i} t_i G_0 \tau_j, \quad (i=1, 2, \cdots A) \quad (18a)$$

then, as is easily seen,

$$t^B = \sum \tau_i. \quad (18b)$$

Assuming [compare with Eq. (10)]

$$t_i = \lambda_i \tau, \quad (19)$$

with the λ_i constants, we set

$$\tau_i = \tau\alpha_i \quad (20)$$

and determine the operators α_i from

$$\alpha_i = \lambda_i + \sum_{j\neq i} \lambda_i Q \alpha_j, \quad (21)$$

obtained by multiplying Eq. (18a) by τ^{-1}. Here

$$Q = G_0\tau. \quad (22)$$

Since Q is the only operator appearing explicitly in Eq. (21), we may treat these equations as ordinary algebraic equations. It follows that

$$\alpha_i = \sum_{j=1}^{A} (B^{-1})_{ij}\lambda_j, \quad (23a)$$

where B is the matrix with elements B_{ij},

$$B_{ii} = 1; \quad B_{ij} = -Q\lambda_j \quad (j\neq i). \quad (23b)$$

Since B is linear in Q, B^{-1} will have poles at $Q=\nu_n$, $n=1, 2, \cdots A$, corresponding to the zeros of det B. In general, these will be simple. Hence, $(B^{-1})_{ij}$ may be written in the form

$$(B^{-1})_{ij} = \sum_{n=1}^{A} \beta_{ij}^n/(\nu_n - Q),$$

where the β_{ij}^n are functions of the λ_κ. It then follows from Eqs. (18), (20), and (23) that t^B may be written in the form

$$t^B = \tau \sum_{n=1}^{A} \gamma_n/(\nu_n - Q).$$

Hence, using Eq. (22), we get

$$t^B = \sum \gamma_n (\tau/\nu_n)[1 - G_0\tau/\nu_n]^{-1},$$

or, recalling Eq. (8),

$$t^B = \sum_{n=1}^{A} \gamma_n T[\tau/\nu_n],$$

so that

$$f^B = \sum_{n=1}^{A} \gamma_n f_n,$$

where f_n is obtained by solving the Schrödinger equation with the pseudopotential τ/ν_n. The quantities γ_n and ν_n are, in general, algebraic functions of the strengths λ_κ. Of course, this method also applies when $A=2$.